MINIMAL UNIVERSAL COVERS IN E^n

BY

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ABSTRACT

It is shown that any plane set of constant unit width contains a semi-circle of radius $\frac{1}{2}$, and using this a minimal univeral plane cover is explicitly constructed. It is also shown that in an n-dimensional spac≥ with $n>2$ there are minimal universal covers of arbitıary large diameter.

Introduction. We shall consider subsets of real n-dimensional Euclidean space E^n. Denote by $\mathscr{K}_n$ that class of subsets of E^n which have a width in every direction equal to 1. If two subsets X and Y of E^n are congruent we write $X \sim Y$.

A subset C of E^n is called a *universal cover* if it is closed, convex and such that for every subset A of E^n whose diameter is less than or equal to 1 we can find a subset B of C such that $B \sim A$. Since every such set A is contained in a member of $\mathscr{K}_n$, it is sufficient in proving that a set C is a universal cover to vertify that the congruent subset B of C can be found corresponding to every set A that belongs to $\mathscr{K}_n$.

By a *minimal* universal cover is meant a universal cover of which no proper subset is also a universal cover.

In the plane the diameter of any minimal cover is less than 3 and the question has been asked (by V. Klee, see [2]) as to whether there is a finite upper bound of the diameters of compact minimal universal covers in E^n, depending possibly on n. We show that this is not the case by proving that for any given positive number K and any integer n with $n \geqq 3$, there is a compact minimal universal cover in E^n whose diameter is greater than K.

We first prove that a certain plane set is a minimal universal plane cover by means of a lemma, which incidentally also shows that any set in $\mathscr{K}_2$ contains a semicircle of radius $\frac{1}{2}$ (see [1]).

§1. **An explicit example of a plane minimal universal cover.** In this section it is shown that a certain set Y, defined explicitly, is a plane minimal universal cover. Y is the union of a disc and of a Reuleaux triangle, both of unit diameter, so

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placed that two of the vertices of the triangle are diametrically opposite points on the disc. See Figure 1.

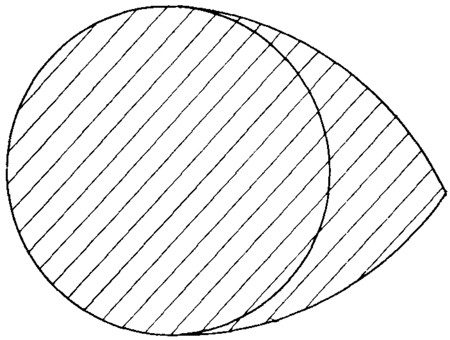

Figure 1

If Y is a universal cover at all it must be a minimal one: indeed no proper subset of Y can contain both a disc and a Reuleaux triangle each of unit diameter. We shall deduce that Y is a universal cover from the following lemma.

LEMMA 1. *If Z is a plane set of unit constant width then there are two points on the frontier of Z, say s,t, which are at unit distance apart and such that of the two semicircles of radius $\frac{1}{2}$ that pass through both s and t, one at least, does not meet the interior of Z.*

For if this lemma were true then, since every point of Z is distant at most 1 from both s and t, it would follow that Z lies in a figure bounded by a semicircle on st and (on the opposite side of st) two arcs of unit radius and centers s and t. This figure is congruent to Y. Thus Z is congruent to a subset of Y. But any set whose diameter does not exceed 1 is contained in a set of unit constant width. Hence Y is a universal cover. It remains to prove the lemma.

Proof of the lemma. By a standard approximation argument it is sufficient to establish the lemma when Z is a Reuleaux polygon and in what follows we consider this case only.

Two vertices of Z, say a,b, are said to be *opposite* if and only if they are at unit distance apart. In the frontier of Z lie two circular arcs whose centers are a and b. They lie on one and the same side of the line ab; we call this the *positive* side and the other side will be called the *negative* side. The circumcircle of the part of Z on the negative side of ab will be denoted by γ_{ab}, its radius by r_{ab} (γ_{ab} is a disc).

In any case $r_{ab} \geqq \frac{1}{2}$ and we wish to establish the existence of two opposite vertices a,b, for which $r_{ab} = \frac{1}{2}$. We assume that no such vertices exist and show that this leads to a contradiction.

Any two vertices p,q on the frontier of Z divide this frontier into two arcs.

Moreover, if p and q are not opposite, then one of these two arcs contains no points opposite to p or q. The vertices of Z on this arc other than p or q are said to lie *between* p and q. If p and q are opposite then by the vertices between p and q we mean those that lie on the negative side of the line pq.

Since $r_{pq} > \frac{1}{2}$, there must, for any pair of opposite vertices p,q, be vertices between p and q which lie on γ_{pq}. Define the vertices v,w such that they lie on γ_{pq} and no other vertices between p and v or between q and w lie on γ_{pq}. Let there be $f(p), g(q)$ vertices between p and v and between q and w respectively. Let $h(pq) = \min(f(p), g(q))$ and $\tau = \min_{p,q} h(pq)$. Choose opposite vertices a,b so that $\tau = h(ab)$ and suppose for definiteness that $f(a) = h(ab)$. Then let b_1 be the vertex opposite a adjacent to b and a_1 be the vertex opposite to b_1 adjacent to a.

We consider two cases.

CASE (i) $\tau = 0$.

a_1 lies on γ_{ab} (see Figure 2). b_1 lies outside γ_{ab} and the line joining b_1 to the center of γ_{ab} bisects internally the angle ab_1a_1. Hence γ_{ab} cuts the segment b_1a_1

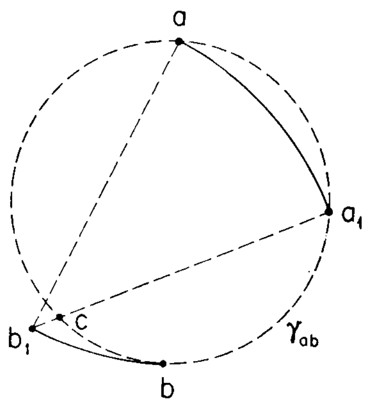

Figure 2

in two points a_1 and c, and the center of γ_{ab} lies on the positive side of a_1b_1. Thus the part of γ_{ab} on the negative side of a_1b_1, apart from the point a_1, lies interior to $\gamma_{a_1b_1}$. But this means that no vertex of Z on the negative side of a_1b_1 lies on $\gamma_{a_1b_1}$. This is impossible since it implies $r_{a_1b_1} = \frac{1}{2}$.

CASE (ii) $\tau > 0$.

Let c be the vertex of Z between a and b on γ_{ab} nearest to a (see Figure 3). Because of the extremal property of a,b and c and because $\tau > 0$, b,c and all the vertices of Z between a_1 and c must be interior points of $\gamma_{a_1b_1}$. Since b,c are interior to $\gamma_{a_1b_1}$, of the two parts of γ_{ab} on the two sides of the line bc one must

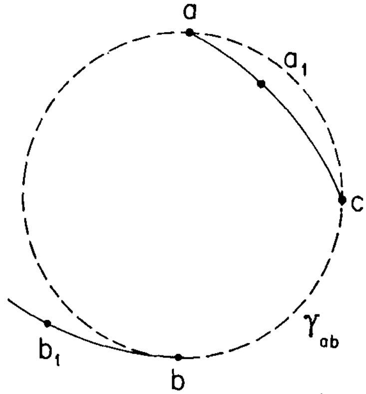

Figure 3

lie interior to $\gamma_{a_1b_1}$. Since a_1 lies on the frontier of $\gamma_{a_1b_1}$ it must be that part of γ_{ab} on the side of bc opposite to a_1. Thus all the vertices of Z between b and c are interior to $\gamma_{a_1b_1}$. Hence all the vertices of Z between a_1 and b_1 are interior to $\gamma_{a_1b_1}$. Again this is impossible.

The lemma 1 is proved.

REMARK. The lemma also shows that any member of K_2 contains a semicircle of radius $\frac{1}{2}$. For if Z is a Reuleaux polygon and in the notation of the lemma $\gamma_{ab} = \frac{1}{2}$ then all vertices of Z on the negative side of ab lie in γ_{ab} and the frontier of Z on the positive side of ab (which is formed from circular arcs of radius 1 with centers at these vertices) lies outside or on the frontier of γ_{ab}. Thus of the frontier of γ_{ab} one semicircle lies inside Z and the other lies outside Z where inside and outside are to be interpreted in the weak sense; i.e., frontier points of Z can lie in the frontier of γ_{ab}.

§2. **A minimal universal cover in E^n of large diameter.** By a *ball* is meant a closed solid sphere; for example, a set such as $\{(x_1,\dots,x_n) \mid \sum_{i=1}^n (x_i - k_i)^2 \leq r^2\}$ where $(x_1,\dots,x_n)$ denote the coordinates of a point in E^n.

We need the following lemma.

LEMMA. 2. *There is a member W of $\mathscr{K}_n$, $n \geq 3$, such that the projection of W in any direction is not an $(n-1)$-dimensional ball.*

Let B_0 be the n-dimensional ball with center $(0,0,\dots,0)$ and radius $\frac{1}{2}$. Let B_i be the ball with radius 1 and center p_i where every coordinate of p_i is zero except the i^{th}, and the i^{th} coordinate is $(7^{\frac{1}{2}} - 1)/2^{\frac{3}{2}}$. Let U be the intersection of B_0 and all the B_i and let V be the union of U and all the points p_i. Let W be a member of $\mathscr{K}_n$ such that $W \supset V$.

Let W_θ be the projection of W in the direction θ. By direct calculation it can be seen that there is a point f on the frontier of W_θ which is the projection of a point of the frontier of W that lies interior to B_0 and on the frontier of one of the B_i. But this means that the frontier of W_θ in a sufficiently small neighborhood of f coincides with the frontier of an $(n-1)$ dimensional ball of radius 1. Thus W_θ is not an $(n-1)$ dimensional of radius $\frac{1}{2}$. Since $W_\theta \in \mathscr{K}_{n-1}$ it follows that W_θ is not an $(n-1)$-dimensional ball.

COROLLARY. *There is a member W of $\mathscr{K}_n$ such that the greatest lower bound of the circumradii of projections of W in all directions is greater than $\frac{1}{2}$.*

The proof is obvious by compactness arguments.

LEMMA 3. *If $Z \in K_2$ and every square circumscribing Z has its sides bisected by the points of contact with Z then Z is a circle.*

Let a, b be two points of Z distant 1 apart and such that the semicircle t_{ab} joining them lies in Z. Let the center of this semicircle be p, and let q be one of the points of Z most distant from p. The line through q perpendicular to pq is a support line of Z: hence by the hypothesis the support lines of Z parallel to pq touch the circle of which t_{ab} is a part. But then by hypothesis pq must be equal to $\frac{1}{2}$. Hence Z is a circle.

Denote by Z that member of $\mathscr{K}_n$ such that the greatest lower bound of the circumradii of projections of Z in all directions has the largest possible value. Let this value be R; then $R > \frac{1}{2}$.

Construction of a universal minimal cover. Let P be the infinite prism $|x_i| \leqq \frac{1}{2}, i = 1,2,\ldots,n-1,\ x_n \geqq 0$. P is a universal cover and our universal minimal cover will be a subset of P. Denote the projection of any set X in E^n onto $x_n = 0$ by X_0. Then if $Y \in \mathscr{K}_n$ and $Y \subset P$ we have $Y_0 \in \mathscr{K}_{n-1}$ and $Y_0 \subset P_0$ where P_0 is the $(n-1)$-dimensional cube whose faces lie in the intersections of the hyperplanes $x_i = \pm\frac{1}{2},\ i = 1,2,\ldots,n-1$, with $x_n = 0$. Of the 2^{n-1} faces of this cube denote that which lies in $x_1 = \frac{1}{2}$ by F. Y_0 meets F in precisely one point; denote this point by $F(Y)$.

Consider the class of sets $\mathscr{Y}$ congruent to Y which lie in P and can be obtained from Y by combining a rotation or reflection which leaves the line $x_1 = x_2 = \ldots = x_{n-1} = 0$ fixed, with a translation perpendicular to this line. Let $F(\mathscr{Y})$ be the set of points formed by $F(Y)$ for all $Y \in \mathscr{Y}$. Let $F^*(\mathscr{Y})$ be the subset of the points of $F(\mathscr{Y})$ that are most distant from the point $(\frac{1}{2}, 0, \ldots, 0)$ (i.e. the center of face F). Both $F(\mathscr{Y})$ and $F^*(\mathscr{Y})$ are closed non-void sets. If $(\frac{1}{2}, x_2, x_3, \ldots, x_{n-1}, 0)$ belongs to $F(\mathscr{Y})$ so do all the 2^{n-2} points $(\frac{1}{2}, \pm x_2, \pm x_3, \ldots, \pm x_{n-1}, 0)$ and they are all the same distance from $(\frac{1}{2}, 0, 0, \ldots, 0)$. Thus there is a point, say $(\frac{1}{2}, x_2^*, x_3^*, \ldots, x_{n-1}^*, 0)$, of $F^*(\mathscr{Y})$ for which $x_i^* \geqq 0$. Select one such point and let Y^* be the set (or one of the sets) belonging to $\mathscr{Y}^*$ such that $F(Y^*) = (\frac{1}{2}, x_2^*, x_3^*, \ldots, x_{n-1}^*, 0)$. It follows from lemma 3 that Y^* is the point $(\frac{1}{2}, 0, 0, \ldots, 0)$ if and only if Y is a ball.

Next let g be a large positive number so that $2(R-1/2)\cdot g > K$ (K is the preassigned positive number, R the minimal circumradius of any projection of Z), and let P_g be the intersection of P with the cone

$$x_1^2 + x_2^2 + \dots + x_{n-1}^2 \leqq \frac{(x_n+g)^2}{(2g+1)^2-1} \tag{1}$$

This cone intersects $x_n = h$ in the $(n-1)$-dimensional ball with center $(0,0,\dots,0,h)$ and radius $r_h = (h+g)/((2g+1)^2-1)^{1/2}$. Since this radius is large for large values of h it follows that P_g is a universal cover.

Also the set P_g contains the n-dimensional unit ball with center at $x_n = \frac{1}{2}, x_i = 0,\ i = 1,\dots,n-1$. Denote this ball by B^*.

For each set Y of $\mathscr{K}_n$ in P select Y^* as described above and translate Y^* parallel to the x_n axis until it lies inside P_g and, subject to this condition, is as close to $x_n = 0$ as possible. Let the translated set be Y^{**}. Let Q be the union of all these sets and let S be the convex cover of the closure of Q. Q and (therefore) S are bounded. S is a subset of P_g that meets the hyperplane $x_n = 0$ (since $Q \supset B^*$) and S also meets the hyperplane $x_n = K$. For consider where Z^{**} can lie. If Z^{**} lies in $x_n < K$ then $R \leqq r_K$, i.e.

$$R \leqq (K+g)/((2g+1)^2-1)^{1/2} ,$$

which implies

$$R \leqq \frac{K+g}{2(g(g+1))^{1/2}} \leqq \frac{K}{2g} + \frac{1}{2}.$$

Thus $2(R-\frac{1}{2})g \leqq K$. But this contradicts our choice of g. Thus Z^{**} lies at least partly in $x_n \geqq K$. Hence S lies partly in $x_n \geqq K$, and the diameter of S is at least K.

S is a compact universal cover; let T be a subset of S that is a minimal compact universal cover. By the same argument as that above T contains points in $x_n \geqq K$. Now the line L defined by $x_1 = \frac{1}{2},\ x_2 = x_3 = \dots = x_{n-1} = 0$ meets Q in the single point $q = (\frac{1}{2}, 0, \dots 0, \frac{1}{2})$. Moreover q is the only point of the closure of Q on L. For if this were not so let p in $\bar{Q}$ lie on L, $p \neq q$. Then there exists a sequence of points p_j such that $p_j \in Q$ and $p_j \to q$ as $j \to \infty$. Then p_j belongs to one of the sets of which Q is the union, say $p_j \in Y_j^{**}$. Now Y_j^{**} meets $x_1 = \frac{1}{2}$ in a single point, say y_j^*, and is moreover contained in an n-dimensional ball of radius 1 touching $x_1 = \frac{1}{2}$ at y_j^*. If the distance py_j^* is $\tan\theta_j$, that of pp_j is greater than or equal to $\sec\theta_j - 1$. Since $p_j \to p$ as $j \to \infty$ it follows that $\theta_j \to 0$ as $j \to \infty$ and thus $y_j^* \to p$ as $j \to \infty$. Let the coordinates of y_j^* be $(y_1^{(j)}, y_2^{(j)}, \dots y_n^{(j)})$. Since p has its 2^{nd}, 3^{rd}, ..., $(n-1^{\text{st}})$ coordinates all zero this means that $y_2^{(j)} \to 0, \dots, y_{n-1}^{(j)} \to 0$. Now a subsequence $\{Y_{j_i}\}$ converges to a member, say W, of $\mathscr{K}_n$ and $F^*(W)$ is the single point $(\frac{1}{2}, 0, 0, \dots, 0)$. By the remark concerning

Lemma 3, W must be such that all its two-dimensional projections are circles. Hence W is an n-dimensional ball, i.e., $W \sim B^*$. Now $Y_{j_i}^{**}$ is one of the sets of which Q is the union and thus $Y_{j_i}^{**}$ can not be translated inside P_g parallel to the x_n axis so that it lies nearer to $x_n = 0$. Thus $Y_{j_i}^{**}$ meets the surface of the cone (1). Hence so also does W. Thus in fact W is B^*. But if this is so, $Y_{j_i}^{**}$ converges to B^*. Hence $y_{j_i}^{**}$ converges to q and p is q.

This contradiction establishes that the line L meets $\bar{Q}$ in the single point $q = (\frac{1}{2}, 0, \ldots, 0, \frac{1}{2})$.

Now S it the convex cover of $\bar{Q}$ and $\bar{Q}$ lies in the set $x_1 \leqq \frac{1}{2}$, and if $x_1 = \frac{1}{2}$, then $0 \leqq x_i \leqq \frac{1}{2}$, $i = 2, \ldots, n-1$, and it follows that S cannot meet the line L in any point other than q.T is a universal cover and hence contains an n-dimensional ball of radius $\frac{1}{2}$.

$S \subset P$ and it follows that S meets the line L. Thus T contains B^*. Thus T contains the point $(0, \ldots, 0)$. Hence T has diameter at least equal to K.

This establishes the required result.

References

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